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The convergence rate of continued fractions representing solutions of a Riccati equation

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Abstract

The aim of this note is to generalize and apply results on matrix continued fractions representing the solution of discrete matrix Riccati equations. Assuming uniform bounds for the norm of the matrix coefficients of the continued fraction, the minimal and maximal solutions of the corresponding algebraic Riccati equation can be accurately enclosed.

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1. Introduction

Continuous fractions have been under investigation for more than a century. The coefficients may not only be complex numbers, but also vectors, matrices or elements of a Banach algebra $\mathfrak{B}$, and these extensions have been applied successfully in various mathematical, physical and control problems.

An important part in the framework of control problems is played by matrix and vector differential equations with a square nonlinearity. Thus, we have developed and implemented algorithms for the verified calculation of solutions of continuous and discrete-time algebraic Riccati equations [5,6]. In these algorithms we can also handle uncertain data, and we get guaranteed enclosures of the exact solution. The Floquet theory deals with a homogeneous vector valued recurrence relation determining the Floquet eigenvalues and eigensolutions. In [8] two solution methods for this recurrence relation are developed based on MCFs. Verified inclusions for eigenvalues and solutions for the underlying boundary value problem of the first-order phase locked loop equation with general phase detector characteristics are obtained in [4]. Further applications concern delay differential equations [10] or vector and matrix Padé approximation [11]. There is a rich literature about MCFs and the discrete matrix Riccati equation (DMRE) [2,3]. However, the problem of verified computation of MCFs and three-term recurrence relations (TTRRs) has rarely been addressed in the past. Recently, Raissouli and Kacha [9] have published some criteria on the convergence of MCFS, but their results include no explicit bounds and are less general that the results given by Otten [7,8].

In this note we show that bounds for the convergence rate of its approximants can be used for verified computations of MCFs representing minimal and maximal solutions of the DMRE. First, we repeat important results such as Pincherle's

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theorem [1] connecting the convergence of MCFs and the existence of a recessive solution of a TTRR. We consider the DMRE $W_{n+1} = A_n + W_n(W_n + C_{n-1})^{-1}C_{n-1}$ and show that important results on the verified calculation of MCFs and linear difference equations carry over to the Riccati context. If we assume uniform bounds for the norm of the matrix coefficients of the continued fraction, then the minimum and maximum solution of the corresponding algebraic Riccati equation can be accurately enclosed.

2. Important results on matrix continued fractions

Assume that the $2n \times 2n$ matrices $\mathbf{A}_k$ defined by $n \times n$ block matrices A_k, B_k, C_k, D_k with real or complex entries are nonsingular. We want to define a sequence of approximants and introduce $P_{m-1} = S_{m-1} = I$, $Q_{m-1} = R_{m-1} = 0$, and for $k \geqslant m$

$$\begin{pmatrix} P_k & Q_k \\ R_k & S_k \end{pmatrix} := \mathbf{A}_m \cdot \mathbf{A}_{m+1} \cdot \ldots \cdot \mathbf{A}_k = \begin{pmatrix} P_{k-1} & Q_{k-1} \\ R_{k-1} & S_{k-1} \end{pmatrix} \mathbf{A}_k.$$

We formally put $T_{\mathbf{A}_m \mathbf{A}_{m+1} \cdot \ldots \cdot \mathbf{A}_k}(\infty) = P_k R_k^{-1}$, $k = m, m+1, \ldots$, where the nonlinear recurrence relation is defined by

$$Z_k = T_{\mathbf{A}_k}(Z_{k+1}) = (A_k Z_{k+1} + B_k)(C_k Z_{k+1} + D_k)^{-1}$$

and compute Z_m following the recurrence relation back from $Z_{k+1} = \infty$, $k \geqslant m$. The continued fraction $T_{\mathbf{A}_m \mathbf{A}_{m+1} \cdot \ldots \cdot \mathbf{A}_k}(\infty)$ is said to be convergent if the denominators R_k are nonsingular for large k and the sequence of approximants $(P_k R_k^{-1})$ has a limit $\Psi(m)$ as $k \to \infty$.

Now we connect the theory of MCFs with the existence of a recessive solution of a certain linear recurrence as presented in [1]. We introduce the transpose or conjugate transpose of the matrices $\mathbf{A}_k$ as $\mathbf{M}(k) := \mathbf{A}_k^{\mathrm{T}}$, $E_k := A_k^{\mathrm{T}}$, $F_k := C_k^{\mathrm{T}}$, $G_k := B_k^{\mathrm{T}}$, $H_k := D_k^{\mathrm{T}}$ and the $2n \times n$ matrix $\mathbf{X}(k) = (Y(k), Z(k))^{\mathrm{T}}$.

Then the pairs $(Y_i(k), Z_i(k))$ with $(Y_1(k), Z_1(k)) = (P_k^{\mathrm{T}}, Q_k^{\mathrm{T}})$, $(Y_2(k), Z_2(k)) = (R_k^{\mathrm{T}}, S_k^{\mathrm{T}})$ are solutions of the recurrent system $\mathbf{X}(k) = \mathbf{M}(k)\mathbf{X}(k-1)$ fulfilling $Y_1(m-1) = Z_2(m-1) = I$, $Z_1(m-1) = Y_2(m-1) = 0$, and the convergence of our MCF to a matrix $\Psi(m)$ is equivalent to the nonsingularity of $Y_2(k)$ for large k and the convergence of $Y_2^{-1}(k)Y_1(k)$ to $\Psi^{\mathrm{T}}(m)$ as $k \to \infty$.

3. Recessive solutions of three-term recurrence relations

A $2n \times n$ solution $\mathbf{X}_0(k)$ of $\mathbf{X}(k) = \mathbf{M}(k)\mathbf{X}(k-1)$ is said to be recessive at ∞, if $\mathbf{X}_0(k)$ has rank n and for another solution $\mathbf{X}(k)$ of the recurrence relation with a nonsingular matrix $[\mathbf{X}, \mathbf{X}_0]$ it follows that $Y(k)$ is nonsingular for large k and $Y^{-1}(k)Y_0(k) \to 0$ as $k \to \infty$. Now we cite the matrix form of Pincherle's theorem due to Ahlbrandt [1]:

Theorem 3.1. *Assume that m is fixed and $\mathbf{A}_k$ are nonsingular matrices for $k = m, m+1, \ldots$. A necessary and sufficient condition for the convergence of the matrix continued fraction $T_{\mathbf{A}_m \mathbf{A}_{m+1} \cdot \ldots \cdot \mathbf{A}_k}(\infty) = P_k R_k^{-1}$, $k = m, m+1 \ldots$, is that there exists a recessive $2n \times n$ solution at ∞, $\mathbf{X}_0(k)$ of $\mathbf{X}(k) = \mathbf{M}(k)\mathbf{X}(k-1)$ with $\mathbf{X}_0 = (Y_0, Z_0)^{\mathrm{T}}$ and $Y_0(m-1)$ nonsingular. If the matrix continued fraction converges to $\Psi(m)$, then $\Psi(m)^{\mathrm{T}} = -Z_0(m-1)Y_0(m-1)^{-1}$.*

Recessive solutions at ∞ with $Y_0(m-1)$ nonsingular are unique up to a nonsingular constant $n \times n$ matrix K, i.e., if $\mathbf{X}_0$ is recessive and K is nonsingular then $\mathbf{X}_0 K$ is recessive. If $\mathbf{X}_0$ and $\mathbf{X}_0'$ are recessive and Y_0 and Y_0' are nonsingular, then there exists a nonsingular matrix K such that $\mathbf{X}_0' = \mathbf{X}_0 K$.

Now we start with $P_{m-1} = I$, $R_{m-1} = P_m = 0$, $R_m = C_m$, and discuss the special case $A_k \equiv 0$. Then, $Y(k) = F_k Z(k-1)$, $Z(k) = G_k Y(k-1) + H_k Z(k-1)$. Replacing $Z(k)$ in the last relation, we find $F_{k+1}^{-1} Y(k+1) = G_k Y(k-1) + H_k F_k^{-1} Y(k)$. Taking the transpose on both sides gives the TTRR for R_k.

In the context of standard right matrix continuous fractions

$$\Psi(0) = \lim_{k \to \infty} A_k B_k^{-1} = \lim_{k \to \infty} b_0 + \underline{a_1}|\overline{b_1 + \underline{a_2}|\overline{b_2 + \cdots + \underline{a_k}|\overline{b_k}}},$$

we put $Y_1(k) := A^{\mathrm{T}}_{k-1}$, $Y_2(k) := B^{\mathrm{T}}_{k-1}$, $Z_1(k) := A^{\mathrm{T}}_k$, $Z_2(k) := B^{\mathrm{T}}_k$. Then it holds $Y_i(k) = Z_i(k-1)$, $i = 1, 2$, with the initial conditions $Y_1(0) = A^{\mathrm{T}}_{-1} = I$, $Y_2(0) = B^{\mathrm{T}}_{-1} = 0$, and $Z_1(0) = A^{\mathrm{T}}_0 = b^{\mathrm{T}}_0$, $Z_2(0) = B^{\mathrm{T}}_0 = 1$, and we are led to the recurrent system

$$A^{\mathrm{T}}_k = a^{\mathrm{T}}_k A^{\mathrm{T}}_{k-2} + b^{\mathrm{T}}_k A^{\mathrm{T}}_{k-1}, \quad B^{\mathrm{T}}_k = a^{\mathrm{T}}_k B^{\mathrm{T}}_{k-2} + b^{\mathrm{T}}_k B^{\mathrm{T}}_{k-1}.$$

The corresponding three-term recurrence is $Y(k+1) = a^{\mathrm{T}}_k Y(k-1) + b^{\mathrm{T}}_k Y(k)$. Thus, Pincherle's theorem is established for this general family of matrix continued fractions. If we want to compute verified inclusion for the approximants then it is necessary to derive bounds for the truncation error.

Next, consider the DMRE $W_{n+1} = A_n + W_n(W_n + C_{n-1})^{-1}C_{n-1}$. Notice that there is a scaling property: if $\|A_n^{-1}\| \leqslant \alpha$, $\|C_n\| \leqslant \gamma$, $\alpha\gamma < 0.25$, where $\|\cdot\|$ denotes the spectral norm, then $\tilde{W}_{n+1} = \tilde{A}_n + \tilde{W}_n(\tilde{W}_n + \tilde{C}_{n-1})^{-1}\tilde{C}_{n-1}$, $\tilde{W}_n := \beta W_n$, $\tilde{A}_n := \beta A_n$, $\tilde{C}_n := \beta C_n$, and it holds $\|\tilde{A}_n^{-1}\| \leqslant \alpha/\beta$, $\|\tilde{C}_n\| \leqslant \beta\gamma$. Choosing $0 < \beta := \sqrt{\alpha/\gamma}$, the norms of the matrices are bounded by the same constant. Thus, Theorem 3 in [9] is a special case of Theorem 5.1 given below.

There is a connection with the TTRR

$$-C_n X_{n+1} - C_{n-1}X_{n-1} + (C_n + C_{n-1} + A_n)X_n = 0 \tag{1}$$

and real symmetric matrices A_n, C_n in the following way. Solutions X_n and W_n are related by $W_n = C_{n-1}(X_n - X_{n-1})X_{n-1}^{-1}$. For two solutions U_n and V_n of the recurrence relation define the bracket symbol

$$\{U_n, V_n\} := U^{\mathrm{T}}_{n-1}C_{n-1}V_n - U^{\mathrm{T}}_n C_{n-1}V_{n-1}.$$

Then $\{U_n, V_n\}^{\mathrm{T}} = -\{V_n, U_n\}$ and $\{U_n, V_n\}$ does not depend on n. A solution X_n is called *prepared* if $\{X_n, X_n\} = 0$.

4. Continued fraction representation of extremum solutions of a discrete matrix Riccati equation

Theorem 4.1 (*Ahlbrandt [2]*). *Suppose that A_n and C_n are positive definite for all $n \in (-\infty, \infty)$. Then, with $D_n = C_n^{-1}$ for any n the approximants*

$$W_n^+(m) := A_{n-1} + \frac{I}{D_{n-2_+}}\frac{I}{A_{n-2_{+\cdots+}}}\frac{I}{A_{m+1_+}}\frac{I}{D_m}$$

converges monotonically from above to the maximal solution W_n^+ as $m \to -\infty$. Furthermore, the approximants

$$W_n^-(m) := D_{n-1} + \frac{I}{A_{n_+}}\frac{I}{D_{n_{+\cdots+}}}\frac{I}{A_{m-1_+}}\frac{I}{D_{m-1}}$$

converge monotonically from below to $-(W_n^-)^{-1}$, as $m \to \infty$, where W_n^- is the unique minimal solution of the Riccati matrix equation.

Under the assumption in Ahlbrandt's theorem there is no difference between a left and a right MCF since the symmetric left and right approximants $B_k^{-1}A_k$ and $A_k B_k^{-1}$ are equal. Pincherle's theorem now states that there is a recessive solution at ∞, Y_n fulfilling the TTRR : $-C_n X_{n+1} - C_{n-1}X_{n-1} + (C_n + C_{n-1} + A_n)X_n = 0$ with $Y_0 = I$ and $-C_0(Y_1 - Y_0) = -W_1^-$.

When A and C are positive definite constant matrices, W_n^+ and W_n^- are of period 1, hence constant. Then it holds that

$$W^+ = \lim A + \frac{I}{D_+}\frac{I}{A_+}\frac{I}{D_{+\cdots+}}\frac{I}{D}, \quad -W^- = W^+ - A$$

and the corresponding maximal solution W^+ is determined by the reverse MCF. If W is any Hermitian solution of $W = A + W[W + C]^{-1}C$ or its equivalent form $WC^{-1}W - AC^{-1}W - A = 0$, then $-C < W^- \leqslant W \leqslant W^+ < A + C$. Also, W^+ and W^- are the only solutions which are positive definite and negative definite, respectively.

5. A bound for the truncation error of matrix continued fractions

Now we can utilize theorems by Otten published in [8] and in more complete form in his thesis [7]. He introduces the Banach-algebra $\mathscr{B}$ and reinterprets the discussion of matrix continued fractions in the context of a normed algebra. He proves the following results:

Theorem 5.1. *Given the sequences $(a_k)_{k=1}^{\infty} \subset \mathscr{B}$ and $(b_k)_{k=1}^{\infty} \subset \mathscr{B}$ and assuming that a_k and b_k are invertible, if the bounds $\|a_k b_k^{-1}\| \leqslant q_1$, $\|b_k^{-1}\| \leqslant q_2$, $q_1 q_2 < 0.25$, hold for all k, then the right approximants $A_k B_k^{-1}$ of the continued fraction*

$$\lim_{k\to\infty} b_0 + \underline{a_1}|\overline{b_1 + \underline{a_2}|\overline{b_2 + \cdots + \underline{a_k}|\overline{b_k}}},$$

converge to a limit $K \in \mathscr{B}$, and K fulfils the following estimation:

$$\|K\| < \|b_0\| + \frac{q_1(1-2q_1q_2)^2}{(1-2q_1q_2)^2 - q_1q_2}.$$

Proof (Sketch). To prove the theorem, we use the relation

$$A_{k-1}B_{k-1}^{-1} - A_k B_k^{-1} = (-1)^k \left(\prod_{v=1}^{k} a_v B_v^{-1} B_{v-1}\right) B_{k-1}^{-1}, \quad k > 0,$$

we put $q := q_2/(1-2q_1q_2)$ and derive the following bounds:

$$\|B_k^{-1}B_{k-1}\| < 2q_2, \quad \|B_k^{-1}\| < q_2 q^{k-1}, \quad \|a_k B_k^{-1} B_{k-1}\| < q_1 q/q_2, \quad k > 0.$$

We can apply this theorem to our MCF coming from Ahlbrand's Theorem 4.1. This enables us to calculate a verified inclusion for K and thus for the minimal and maximal solution of the DMRE. Since recessive solutions are unique up to a nonsingular matrix K, the initial value $Y_0 = I$ includes the general case. □

In analogy to the work by Otten it is possible to derive a norm bound for Y_n and verified inclusions for the matrix sequence (Y_n). We can start with $Y_0 = I$ and find $Y_1 = C_0^{-1}W_1^{-} + I$. Y_n is calculated via the TTRR

$$Y_n = (I + C_n^{-1}C_{n+1} + C_n^{-1}A_{n+1})Y_{n+1} + (-C_n^{-1}C_{n+1})Y_{n+2}.$$

Now we come to our main theorem linking the convergence rate of the recessive solution to assumptions on the bounds of the norms of real and symmetric matrices A_n^{-1} and D_n^{-1}. The result should be compared with the Pincherle convergence theorem (Theorem 13.2 in [2]).

Theorem 5.2. *From the assumption $\|A_k^{-1}\| \leqslant q < 0.5$, $\|D_k^{-1}\| \leqslant q$ for all k and $Q := q/(1-2q^2)$ it follows for two solutions (U_n), (V_n) of the TTRR* (1) *with respect to the initial conditions $U_0 = U_1 = I$, $V_0 = 0$, $V_1 = C_0^{-1}$, that*

$$\|V_{n+1}^{-1}U_{n+1} - V_n^{-1}U_n\| < \alpha Q^{4n-4} \text{ for a suitable } \alpha$$

and

$$\|Y_n\| < \beta Q^{2n} \text{ for a suitable } \beta.$$

Proof. Define a matrix linear fractional transformation by $T_n(W) = A_n + W(W + C_{n-1})^{-1}C_{n-1}$. It follows that

$$W_n(m) = \underbrace{T_{n-1} \circ T_{n-2} \circ \ldots \circ T_{m+2}}_{T}(A_{m+1} + C_m), \quad n = m+2, \ldots,$$

where T is given by $T(Z) = (EZ + F)(GZ + H)^{-1}$ with

$$E = C_{n-1}(V_n - V_{n-1}), \quad F = C_{n-1}(U_n - U_{n-1}), \quad G = V_{n-1}, \quad H = U_{n-1}.$$

Here U_n and V_n are solutions of $-C_n X_{n+1} - C_{n-1} X_{n-1} + (C_n + C_{n-1} + A_n) X_n = 0$ with initial conditions $U_{m+1} = I = U_{m+2}$, $V_{m+1} = 0$, $V_{m+2} = C_{m+1}^{-1}$. Furthermore, U_n and V_n are prepared [2].

There are several relations between the numerators U_n and the denominators V_n, especially coming from the bracket relation $U_{n-1}^{\mathrm{T}} C_{n-1} V_n V_{n-1}^{-1} - U_n^{\mathrm{T}} C_{n-1} = V_{n-1}^{-1}$, and since V_n is prepared it holds that $U_{n-1}^{\mathrm{T}} V_{n-1}^{-\mathrm{T}} - U_n^{\mathrm{T}} V_n^{-\mathrm{T}} = V_{n-1}^{-1} C_{n-1}^{-1} V_{n-1}^{-\mathrm{T}}$.

Transposing the last relation yields for the series of approximants $S_n(0) := V_n^{-1} U_n$ with respect to the initial conditions $U_0 = U_1 = I$, $V_0 = 0$, $V_1 = C_0^{-1}$,

$$S_{n+p}(0) - S_n(0) = \sum_{k=n}^{n+p-1} (V_{k+1}^{\mathrm{T}} C_k V_k)^{-1}. \tag{2}$$

Using Theorem 5.1 we obtain

$$U_k = A_{2k-1}^{\mathrm{T}}, \quad V_k = B_{2k-1}, \quad V_k = V_k^{\mathrm{T}},$$

$$\|V_k^{-1}\| \leqslant q^{2k-1}/(1-2q^2)^{2k-2}, \quad \|V_k^{-1} V_{k-1}\| < 4q^2,$$

$$\|B_k^{-1} B_{k-1}\| < 2q, \quad V_{k-1} C_k^{-1} = B_{2k-3} C_k^{-1} = B_{2k-2} - B_{2k-4}$$

and with $\alpha = 2(1-2q^2)^6/((1-2q^2)^4 - q^4)$ it follows that

$$\begin{aligned} \|S_{n+p}(0) - S_n(0)\| &\leqslant \left\| \sum_{k=n}^{n+p-1} V_k^{-1} V_{k-1}^{-1} V_{k-1} C_k^{-1} V_{k+1}^{-\mathrm{T}} \right\| \\ &< \sum_{k=n}^{n+p-1} \|V_k^{-1}\| \cdot \|V_{k+1}^{-\mathrm{T}}\| + \sum_{k=n}^{n+p-1} \|V_k^{-1}\| \cdot \|V_{k-1}^{-1}\| \\ &\leqslant 2 \sum_{k=n}^{n+p-1} \frac{q^{4k-4}}{(1-2q^2)^{4k-6}} \leqslant \alpha \frac{q^{4n-4}}{(1-2q^2)^{4n-4}}. \end{aligned}$$

Now it is possible to derive a norm bound for Y_n and to obtain verified inclusions for the matrix sequence (Y_n) by using interval computations. We can start with $Y_0 = I$ and find $Y_1 = C_0^{-1} W_1^- + I$, $Y_n = U_n - V_n(-W_1^-) = U_n - V_n \Psi$. We show that Y_n is recessive and argue as above. From (1) and (2) and using $V_n V_k^{-1} C_k^{-1} V_{k+1}^{-\mathrm{T}} = V_n V_{n+1}^{-1} V_{n+1} \cdots V_k V_k^{-1} V_{k-1}^{-1} (B_{2k-2} - B_{2k-4}) V_{k+1}^{-1}$ we obtain

$$\begin{aligned} Y_n &= V_n \sum_{k=n}^{\infty} (V_{k+1}^{\mathrm{T}} C_k V_k)^{-1}, \\ \|Y_n\| &\leqslant \left\| V_n \sum_{k=n}^{\infty} V_k^{-1} C_k^{-1} V_{k+1}^{-\mathrm{T}} \right\| < 4q \sum_{k=n}^{\infty} (4q^2)^{k+1-n} \|V_{k-1}^{-1}\| \\ &\leqslant 16 \frac{(1-2q^2)^6}{(1-2q^2)^2 - 4q^4} \frac{q^{2n}}{(1-2q^2)^{2n}} = \beta Q^{2n}. \end{aligned}$$

Even in the general case without assuming norm bounds for the matrices $\|D_k^{-1}\|$, $\|A_k^{-1}\|$ it can be shown that $X_n^{-1} Y_n$ tends to the zero-matrix as $n \to \infty$. $\square$

6. Conclusion

The relationship between matrix continued fractions, the TTRR, and the discrete matrix Riccati equation has been pointed out and a convergence rate for MCF has been derived. We have shown that verified calculations for extremum solutions of the discrete matrix Riccati equation can be achieved. Moreover, the TTRR can be solved with result verification.

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